1.a) (1.7-1)

Note: you only need to show $H\{\alpha x\} \neq \alpha H\{x\}$ OR $H\{x_1+x_2\} \neq H\{x_1\} + H\{x_2\}$. I show both for completeness.

$$\frac{dy}{dt} + 2y = x^2$$

an input of αx ($\alpha \in \mathbb{C}$ is constant) gives

$$(\alpha x)^2 = \alpha^2 x^2 = \alpha^2\left(\frac{dy}{dt} + 2y\right) = \frac{d(\alpha^2 y)}{dt} + 2(\alpha^2 y) \quad \text{(derivatives are linear)}$$

an input of αx gives output of $\alpha^2 y \rightarrow$ $\boxed{\text{nonlinear}}$

y_1 and y_2 are outputs for x_1 and x_2, respectively. the sum output gives

$$\frac{d}{dt}(y_1+y_2) + 2(y_1+y_2) = \frac{dy_1}{dt} + \frac{dy_2}{dt} + 2y_1 + 2y_2$$

$$= \left(\frac{dy_1}{dt} + 2y_1\right) + \left(\frac{dy_2}{dt} + 2y_2\right) = x_1^2 + x_2^2 = (x_1+x_2)^2 - 2x_1x_2$$

$$\neq (x_1+x_2)^2 \rightarrow \boxed{\text{nonlinear}}$$

1.b

Note: You must show both superposition ($H\{x_1\} + H\{x_2\} = H\{x_1+x_2\}$) AND scaling ($H\{\alpha x\} = \alpha H\{x\}$) hold.

$$\frac{dy}{dt} + 3ty = t^2 x$$

$$\frac{d(\alpha y)}{dt} + 3t(\alpha y) = \alpha\frac{dy}{dt} + \alpha 3ty = \alpha\left(\frac{dy}{dt} + 3ty\right) = \alpha(t^2x) = t^2(\alpha x)$$

$$\frac{d}{dt}(y_1+y_2) + 3t(y_1+y_2) = \frac{dy_1}{dt} + \frac{dy_2}{dt} + 3ty_1 + 3ty_2 \quad \boxed{\text{linear}}$$

$$= \left(\frac{dy_1}{dt} + 3ty_1\right) + \left(\frac{dy_2}{dt} + 3ty_2\right) = (t^2x_1) + (t^2x_2) = t^2(x_1+x_2)$$

$\boxed{1.c}$

$$3y + 2 = x$$

$$3(\alpha y) + 2 \neq \alpha(3y + 2) = \alpha x \quad \boxed{\text{non linear}}$$

$$3(y_1 + y_2) + 2 = (3y_1) + (3y_2 + 2) \neq (3y_1 + 2) + (3y_2 + 2) = x_1 + x_2$$

$\boxed{1.d}$ $$\frac{dy}{dt} + y^2 = x \quad \boxed{\text{non linear}}$$

$$\frac{d}{dt}(\alpha y) + (\alpha y)^2 = \alpha\frac{dy}{dt} + \alpha^2 y^2 = \alpha\left(\frac{dy}{dt} + \alpha y^2\right) \neq \alpha\left(\frac{dy}{dt} + y^2\right) = \alpha x$$

$$\frac{d}{dt}(y_1 + y_2) + (y_1 + y_2)^2 = \frac{dy_1}{dt} + \frac{dy_2}{dt} + y_1^2 + y_2^2 + 2y_1y_2$$

$$= \left(\frac{dy_1}{dt} + y_1^2\right) + \left(\frac{dy_2}{dt} + y_2^2\right) + 2y_1y_2 \neq \left(\frac{dy_1}{dt} + y_1^2\right) + \left(\frac{dy_2}{dt} + y_2^2\right) = x_1 + x_2$$

$\boxed{1.e}$ $$\left(\frac{dy}{dt}\right)^2 + 2y = x \quad \boxed{\text{non linear}}$$

$$\left(\frac{d}{dt}(y_1 + y_2)\right)^2 + 2(y_1 + y_2) = \left(\frac{dy_1}{dt} + \frac{dy_2}{dt}\right)^2 + 2y_1 + 2y_2$$

$$= \left(\left(\frac{dy_1}{dt}\right)^2 + 2y_1\right) + \left(\left(\frac{dy_2}{dt}\right)^2 + 2y_2\right) + 2\frac{dy_1}{dt}\frac{dy_2}{dt}$$

$$\neq \left(\left(\frac{dy_1}{dt}\right)^2 + 2y_1\right) + \left(\left(\frac{dy_2}{dt}\right)^2 + 2y_2\right) = x_1 + x_2$$

$$\left(\frac{d(\alpha y)}{dt}\right)^2 + 2(\alpha y) = \left(\alpha\frac{dy}{dt}\right)^2 + 2\alpha y = \alpha^2\left(\frac{dy}{dt}\right)^2 + \alpha 2y$$

$$= \alpha\left(\alpha\left(\frac{dy}{dt}\right)^2 + 2y\right) \neq \alpha\left(\left(\frac{dy}{dt}\right)^2 + 2y\right) = \alpha x$$

$\boxed{1.f}$ $\frac{dy}{dt} + \sin(t)\, y = \frac{dx}{dt} + 2x$ $\boxed{\text{linear}}$

$$\frac{d(y_1+y_2)}{dt} + \sin(t)(y_1+y_2) = \frac{dy_1}{dt} + \frac{dy_2}{dt} + \sin(t)y_1 + \sin(t)y_2$$

$$= \left(\frac{dy_1}{dt} + \sin(t)y_1\right) + \left(\frac{dy_2}{dt} + \sin(t)y_2\right) = \left(\frac{dx_1}{dt} + 2x_1\right) + \left(\frac{dx_2}{dt} + 2x_2\right)$$

$$= \left(\frac{dx_1}{dt} + \frac{dx_2}{dt}\right) + (2x_1 + 2x_2) = \frac{d}{dt}(x_1+x_2) + 2(x_1+x_2)$$

$$\frac{d(\alpha y)}{dt} + \sin(t)(\alpha y) = \alpha\frac{dy}{dt} + \alpha\sin(t)y = \alpha\left(\frac{dy}{dt} + \sin(t)y\right)$$

$$= \alpha\left(\frac{dx}{dt} + 2x\right) = \alpha\frac{dx}{dt} + \alpha 2x = \frac{d(\alpha x)}{dt} + 2(\alpha x)$$

$\boxed{1.g}$ $\frac{dy}{dt} + 2y = x\frac{dx}{dt}$ $\boxed{\text{non linear}}$

$$\frac{d}{dt}(y_1+y_2) + 2(y_1+y_2) = \left(\frac{dy_1}{dt} + 2y_1\right) + \left(\frac{dy_2}{dt} + 2y_2\right) = x_1\frac{dx_1}{dt} + x_2\frac{dx_2}{dt}$$

$$\neq (x_1+x_2)\frac{d(x_1+x_2)}{dt} = (x_1+x_2)\left(\frac{dx_1}{dt} + \frac{dx_2}{dt}\right)$$

$$= x_1\frac{dx_1}{dt} + x_2\frac{dx_2}{dt} + x_1\frac{dx_2}{dt} + x_2\frac{dx_1}{dt}$$

$$\frac{d(\alpha y)}{dt} + 2(\alpha y_1) = \alpha\left(\frac{dy}{dt} + 2y\right) = \alpha x\frac{dx}{dt} \neq (\alpha x)\frac{d(\alpha x)}{dt} = \alpha^2 x\frac{dx}{dt}$$

$\boxed{1.h}$

$$y = \int_{-\infty}^{t} x \, d\tau \qquad \boxed{\text{linear}}$$

$$y_1 + y_2 = \int_{-\infty}^{t} x_1 \, d\tau + \int_{-\infty}^{t} x_2 \, d\tau = \int_{-\infty}^{t} (x_1 + x_2) \, d\tau$$

$$\alpha y_1 = \alpha \int_{-\infty}^{t} x \, d\tau = \int_{-\infty}^{t} (\alpha x) \, d\tau$$

Aside on linearity: sums and scalar multiplication preserve linearity.
If I have 2 linear systems, H_1 and H_2, then the system $H = \alpha H_1 + \beta H_2$ is also linear! ($c_1, c_2, \alpha, \beta \in \mathbb{C}$ are constants)

$$\begin{aligned}
H\{c_1 x_1 + c_2 x_2\} &= \alpha H_1\{c_1 x_1 + c_2 x_2\} + \beta H_2\{c_1 x_1 + c_2 x_2\} \\
&= \alpha\left(c_1 H_1\{x_1\} + c_2 H_1\{x_2\}\right) + \beta\left(c_1 H_2\{x_1\} + c_2 H_2\{x_2\}\right) \\
&= \alpha c_1 H_1\{x_1\} + \alpha c_2 H_1\{x_2\} + \beta c_1 H_2\{x_1\} + \beta c_2 H_2\{x_2\} \\
&= c_1\left(\alpha H_1\{x_1\} + \beta H_2\{x_1\}\right) + c_2\left(\alpha H_1\{x_2\} + \beta H_2\{x_2\}\right) \\
&= c_1 H\{x_1\} + c_2 H\{x_2\}
\end{aligned}$$

Thus all differential equations of the form

$$c_n \frac{d^n}{dt^n} y + c_{n-1} \frac{d^{n-1}}{dt^{n-1}} y + \cdots + c_1 \frac{d}{dt} y + p(t) y = b_m \frac{d^m}{dt^m} x + b_{m-1} \frac{d^{m-1}}{dt^{m-1}} x + \cdots + b_1 \frac{dx}{dt} + q(t) x$$

are linear because $\frac{d^n}{dt} y = x$ and $p(t) y = x$ are linear. (p + q are functions of t)

Also, when you see a form where the input or output are multiplied by themselves or each other in some way, it should be a BIG RED FLAG that the system is probably not linear.

2.a) (1.7-2)

$y(t) = x(t-2)$

Denote a shifted version of x by $x_\tau(t) = x(t-\tau)$. $y_\tau(t)$ is the output for this input, If $y_\tau(t) = y(t-\tau)$ then the system must be translation invariant (TI). If $y_\tau(t) \neq y(t-\tau)$ then the system must be translation varying (TV).

$$y_\tau(t) = x_\tau(t-2) = x((t-2)-\tau) = x((t-\tau)-2) = y(t-\tau) \quad \boxed{TI}$$

2.b)

$y(t) = x(-t)$

$$y_\tau(t) = x_\tau(-t) = x(-t-\tau) \neq x(-(t-\tau)) = y(t-\tau) \quad \boxed{TV}$$

2.c)

$y(t) = x(at)$

$$y_\tau(t) = x_\tau(at) = x(at-\tau) \neq x(a(t-\tau)) = y(t-\tau) \quad \boxed{TV}$$

2.d)

$y(t) = t\,x(t-2)$

$$y_\tau(t) = t\,x_\tau(t-2) = t\,x((t-2)-\tau) \neq (t-\tau)\,x((t-\tau)-2) = y(t-\tau) \quad \boxed{TV}$$

2.e)

$y(t) = \int_{-5}^{5} x(t)\,dt$ Note: $y(t)$ is a constant

$$y_\tau(t) = \int_{-5}^{5} x_\tau(t)\,dt = \int_{-5}^{5} x(t-\tau)\,dt = \int_{-5-\tau}^{5-\tau} x(u)\,du \neq y(t-\tau) = y(t) \quad \boxed{TV}$$

$u = t-\tau$

$t = u+\tau$

$dt = du$

2.f

$$y(t) = \left(\frac{dx}{dt}\right)^2$$

$$y_\tau(t) = \left(\frac{d}{dt}(x_\tau)\right)^2 = \left(\frac{d}{dt}x(t+\tau)\right)^2 = \underset{\text{(chain rule)}}{\left(x'(t-\tau)\right)^2} = y(t-\tau) \qquad \boxed{\text{TI}}$$

$$x'(t) = \frac{dx}{dt}$$

3.a (1.1-7)

i) $$E\{Tx_1\} = \int_{-\infty}^{\infty} |Tx_1(t)|^2\,dt = \int_{-\infty}^{\infty} T^2 |x_1(t)|^2\,dt = T^2 \int_{-\infty}^{\infty} |x_1|^2\,dt = T^2 E\{x_1\}$$

ii) $$E\{x_1(t-T)\} = \int_{-\infty}^{\infty} |x_1(t-T)|^2\,dt = \int_{-\infty-T}^{\infty-T} |x_1(u)|^2\,du = \int_{-\infty}^{\infty} |x_1(u)|^2\,du = E\{x_1\}$$

iii) x_1 is nonzero only if $x_2 = 0$
x_2 is nonzero only if $x_1 = 0$ $\rightarrow x_1 x_2 = 0 \;\forall t$

$$E\{x_1 + x_2\} = \int_{-\infty}^{\infty} |x_1(t) + x_2(t)|^2\,dt = \int_{-\infty}^{\infty} (x_1+x_2)(x_1+x_2)^*\,dt$$

$$= \int_{-\infty}^{\infty}\Big(x_1x_1^* + x_2x_2^* + \underbrace{x_1x_2^* + \underbrace{x_1^*x_2}_{(x_1x_2^*)^*}}_{2\mathrm{Re}(x_1x_2^*)}\Big)\,dt$$

Note: These signals are orthogonal

$$= \int_{-\infty}^{\infty}\left(|x_1|^2 + |x_2|^2 + 2\mathrm{Re}(x_1x_2^*)\right)dt$$

$$= \int_{-\infty}^{\infty} |x_1|^2\,dt + \int_{-\infty}^{\infty} |x_2|^2\,dt + \int_{-\infty}^{\infty} 2\mathrm{Re}(\underbrace{x_1x_2^*}_{=0\;\forall t})\,dt$$

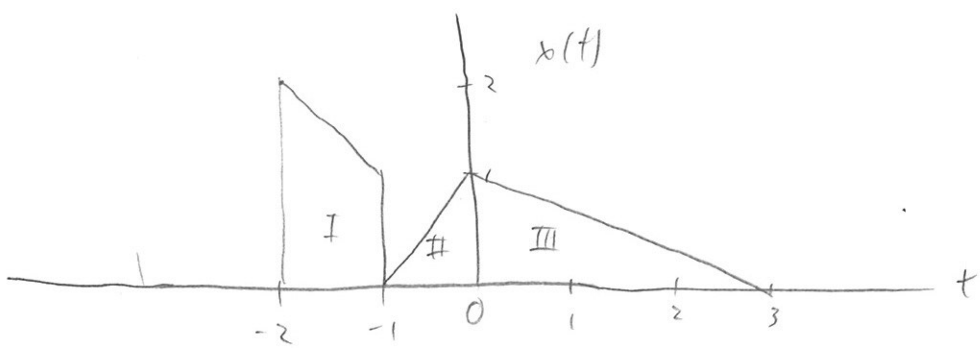
$$= E\{x_1\} + E\{x_2\} + \int_{-\infty}^{\infty} 0\,dt = E\{x_1\} + E\{x_2\}$$

iv) $$E\{x_1(Tt)\} = \int_{-\infty}^{\infty} |x_1(Tt)|^2\,dt = \int_{-\infty T}^{\infty T} |x_1(u)|^2 \frac{du}{T}$$

$$= \begin{cases} \frac{1}{T}\int_{-\infty}^{\infty} |x_1(u)|^2\,du = \frac{1}{T}E\{x_1\} \\ \frac{1}{T}\int_{\infty}^{-\infty} |x_1(u)|^2\,du = -\frac{1}{T}\int_{-\infty}^{\infty} |x_1(u)|^2\,du = -\frac{1}{T}E\{x_1\} \end{cases}$$

$$= \frac{1}{|T|} E\{x_1\}$$

3.b



I: $\int_{-2}^{-1} (-t)^2 \, dt = \frac{1}{3} t^3 \Big]_{t=-2}^{-1} = \frac{1}{3}(1 - -8) = 3$

II: $\int_{-1}^{0} (t+1)^2 \, dt = \frac{1}{3}(t+1)^3 \Big]_{t=-1}^{0} = \frac{1}{3}(1-0) = \frac{1}{3}$

III: $\int_{0}^{3} \left(-\frac{1}{3}(t-3)\right)^2 dt = = \frac{1}{3^2} \cdot \frac{1}{3}(t-3)^3 \Big]_{t=0}^{3} = \frac{1}{3^3}\left(0 - -3^3\right) = 1$

$E\{x\} = 3 + \frac{1}{3} + 1 = \boxed{\frac{13}{3}}$

4.a (1.5-12)

$x(t) = (1+j)t^2$ when $1 \le t \le 2$

$x^*(-t) = -x(t)$ (x is skew (or anti) hermitian)

x is the minimum energy signal with these properties

$x(t) = -\left(\left((1+j)(-t)^2\right)\right)^*$ when $-2 \le t \le -1$ in order for x to be antihermitian

$= (-1+j)t^2$

$E\{x\} = \int_{-\infty}^{\infty} |x|^2 dt = \int_0^{\infty} |x|^2 dt + \int_{-\infty}^{0} |-x^*(-t)|^2 dt$

$|z|^2 = |z^*|^2 = |-z|^2 \quad \forall z \in \mathbb{C}$

$= \int_0^{\infty} |x|^2 dt + -\int_{\infty}^{0} |-x^*(t)|^2 dt = \int_0^{\infty} |x|^2 dt + \int_0^{\infty} |x|^2 dt$

$= 2\int_0^{\infty} |x|^2 dt$ (in other words, $|x|^2$ is even if x is antihermitian)

$= 2\int_0^1 |x|^2 dt + 2\int_1^2 |x|^2 dt + 2\int_2^{\infty} |x|^2 dt$

all of these are ≥ 0, therefore E will be minimized if we set the undefined parts of x to zero

4.a cont

$$E\{x\} = 2\int_1^2 |x|^2 dt$$

$$= 2\int_1^2 |(1+j)t^2|^2 dt = 2|1+j|^2 \int_1^2 t^4 dt$$

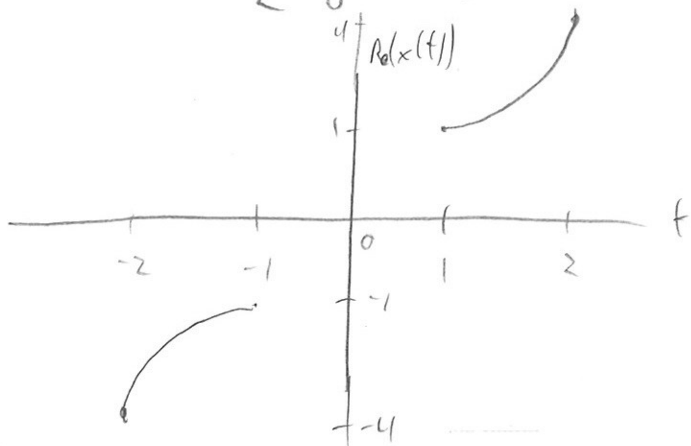
$$= 2 \cdot 2 \left(\tfrac{1}{5} t^5 \Big]_{t=1}^{2} \right) = \frac{4}{5}(2^5 - 1) = \frac{124}{5}$$

$$x(t) = \begin{cases} (1+j)t^2 & 1 \le t \le 2 \\ -(1-j)t^2 & -2 \le t \le -1 \\ 0 & \text{else} \end{cases} \quad \text{or} \quad x(t) = \begin{cases} (1+j)t^2 & 1 \le t \le 2 \\ 0 & 0 \le t \le 1 \text{ or } t > 2 \\ -x^*(-t) & t < 0 \end{cases}$$

4.b

$$\mathrm{Re}(x(t)) = \begin{cases} t^2 & 1 \le t \le 2 \\ -t^2 & -2 \le t \le -1 \\ 0 & \text{else} \end{cases}$$

note: anti hermitian functions have odd real parts

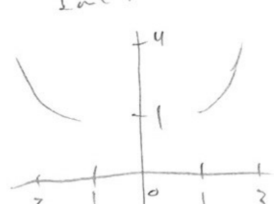


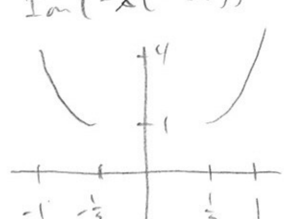
4.c

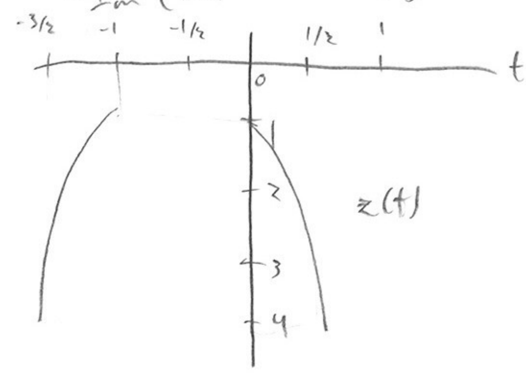
$$z(t) = \mathrm{Re}(jx(-2t+1)) = -\mathrm{Im}(x(-2t+1)) = -\mathrm{Im}(x(-2(t+\tfrac{1}{2})))$$

$$\mathrm{Im}(x(t)) = \begin{cases} t^2 & 1 \le t \le 2 \\ t^2 & -2 \le t \le -1 \\ 0 & \text{else} \end{cases}$$

note: antihermitian functions have even imaginary parts







4.d

$E\{x\} = \boxed{\frac{124}{5}}$ (see part a)

$P\{x\} = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} |x|^2 dt$

$= \lim_{T\to\infty} \frac{1}{T}\frac{124}{5}$ (for $T > 2$)

$= \boxed{0}$

5.a (1.5-3)

even $\cdot$ odd = odd

$x_e(-t)\, x_o(-t) = x_e(t)(-x_o(t)) = -x_e(t)\, x_o(t)$

the integral of an odd function over a symmetric interval centered at zero is zero

$\int_{-a}^{a} (\text{odd})\, dt = 0$

$\int_{-a}^{a} x_o(t)\, dt = \int_{-a}^{0} x_o(t)\, dt + \int_{0}^{a} x_o(t)\, dt = \int_{a}^{0} x_o(-t)(-dt) + \int_{0}^{a} x_o(t)\, dt$

$= \int_{0}^{a} x_o(-t)\, dt + \int_{0}^{a} x_o(t)\, dt = -\int_{0}^{a} x_o(t)\, dt + \int_{0}^{a} x_o(t)\, dt = 0$

Therefore, $\int_{-\infty}^{\infty} x_e(t)\, x_o(t)\, dt = 0$

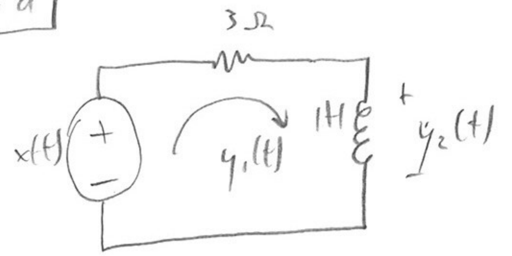
5.b

$x(t) = x_e(t) + x_o(t)$

$\int_{-\infty}^{\infty} x(t)\, dt = \int_{-\infty}^{\infty} (x_e(t) + x_o(t))\, dt = \int_{-\infty}^{\infty} x_e(t)\, dt + \cancel{\int_{-\infty}^{\infty} x_o(t)\, dt}^{\,0}$

$= \int_{-\infty}^{\infty} x_e(t)\, dt = 2\int_{0}^{\infty} x_e(t)\, dt$ (you should know this too)

6. a



$L \frac{di}{dt} = v$

$Ri = v$

$$\boxed{x(t) = 3y_1 + 1\frac{dy_1}{dt}}$$

$$y_2(t) = 1 \cdot \frac{dy_1}{dt}$$

$$\frac{dx}{dt} = 3\frac{dy_1}{dt} + \frac{d^2y_1}{dt^2} = \boxed{3y_2 + \frac{dy_2}{dt} = \frac{dx}{dt}}$$

6.b) From the Note in Problem 1, this is a pair of $\boxed{\text{linear}}$ systems.

$$\frac{dy_{1\tau}}{dt} + 3y_{1\tau} = \frac{d}{dt}(y_1(t-\tau)) + 3y_1(t-\tau) = y_1'(t-\tau) + 3y_1(t-\tau) = x(t-\tau) = x_\tau(t)$$

$y_1' = \frac{dy_1}{dt}$ current is $\boxed{TI}$

$$\frac{dx_\tau}{dt} = \frac{d}{dt}(x(t-\tau)) = x'(t-\tau)$$

voltage is $\boxed{TI}$

Note: derivatives and anti derivatives are TI.